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Elements of an Improved Model of Debris-flow Motion

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Abstract. A new depth-averaged model of debris-flow motion describes simultaneous evolution of flow velocity and depth, solid and fluid volume fractions, and pore-fluid pressure. Non-hydrostatic pore-fluid pressure is produced by dilatancy, a state-dependent property that links the depth-averaged shear rate and volumetric strain rate of the granular phase. Pore-pressure changes caused by shearing allow the model to exhibit rate-dependent flow resistance, despite the fact that the basal shear traction involves only rate-independent Coulomb friction. An analytical solution of simplified model equations shows that the onset of downslope motion can be accelerated or retarded by pore-pressure change, contingent on whether dilatancy is positive or negative. A different analytical solution shows that such effects will likely be muted if downslope motion continues long enough, because dilatancy then evolves toward zero, and volume fractions and pore pressure concurrently evolve toward steady states.

Keywords: debris flow, landslide, granular, mixture, pore pressure, dilatancy, soil mechanics, mathematical model
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INTRODUCTION

Debris flows are geophysical phenomena intermediate in character between rock avalanches and flash floods. They commonly originate as water-laden landslides on steep slopes and transform into liquefied masses of fragmented rock, muddy water, and organic matter that disgorge from canyons onto valley floors. Typically including 50 to 70 percent solid grains by volume, attaining speeds >10 m/s, and ranging in size up to ~10^9 m^3, debris flows can denude slopes, inundate floodplains, and devastate people and property. Notable recent debris-flow disasters resulted in more than 20,000 fatalities in Armero, Colombia in 1985 and in Vargas state, Venezuela in 1999.

Scientific study of debris-flow behavior began more than 100 years ago [1], and experimental and theoretical investigations advanced significantly as the 20th century drew to a close [2-6]. Several groups subsequently developed computational models of debris-flow motion by using depth-averaged, shallow-flow equations generalized from those of the Savage-Hutter model of granular avalanching [7,8]. The most fundamental generalization involved modeling the effects of intergranular liquid, called pore fluid in the geophysical and geotechnical literature [9, 10].

Pore-fluid pressure plays a crucial role in debris flows because it counteracts normal stresses at grain contacts and thereby reduces intergranular friction and enhances bulk flow mobility [6]. Typically, two-phase debris-flow models assume that pore-fluid pressure has both a hydrostatic component and a non-hydrostatic component that is established by initial conditions and dissipated diffusively in response to debris compaction driven by gravity [11]. These models lack a key ingredient, however: explicit evolution of solid and fluid volume fractions coupled to changes in flow dynamics. Such evolution is a fundamental feature of unsteady grain-fluid flows [6]. It is particularly important during the initial stages of debris-flow motion, when it is responsible for pore-pressure feedbacks that influence the balance of forces governing downslope acceleration [12]. As a result of these feedbacks, an initially static mass can either creep stably or mobilize into a high-speed flow [13].

In this paper I summarize the elements of a new, depth-averaged debris-flow model that accounts for coupled evolution of flow dynamics and volume fractions by combining approaches previously used to model landslides [13], debris flows [11], submarine granular avalanches [14], and other dense granular flows [15]. The model's structure is also consistent with a long-established tenet of critical-state soil mechanics [16]: solid and fluid volume fractions evolve toward values that are equilibrated to the ambient state of effective stress and deformation. Dilatancy, pore pressure, and frictional resistance evolve as a consequence.

CONCEPTUAL FRAMEWORK

To emphasize physical concepts and minimize mathematical complexity, I restrict attention to one-dimensional motion of a two-dimensional debris flow descending a rigid, impermeable plane uniformly inclined at the angle θ (Figure 1). The flow moves downslope as an evolving surge that has a characteristic length, L, characteristic thickness, H, and characteristic grain diameter, δ. The fact that the flow has these three length scales, and that $L >> H >> \delta$ is typical, plays an important role in model formulation.

The model assumes that the debris consists of incompressible solid grains of mass density ρ_s occupying the volume fraction m mixed with incompressible fluid of mass density ρ_f occupying the volume fraction $1-m$, such that the mixture bulk density is:

$$\rho = \rho_s m + \rho_f (1-m) \quad . \tag{1}$$

The depth-averaged value of m is a dependent variable in the model, although much of the theoretical development is applicable even if m is a function of y. The other dependent variables are v, the depth-averaged flow velocity in the x (downslope) direction, and h, the flow thickness in the y direction. Additional quantities that evolve in the model do so as specified functions of m, v, and h.

The model focuses on the macroscopic, depth-averaged mechanics of a debris flow as a whole, but it relates macroscopic behavior to grain-scale behavior in a rudimentary way through use of a dimensionless state parameter. This parameter, originally identified by Savage [17] and later dubbed the Savage number [6], represents a ratio of grain-scale inertial stresses caused by shearing to bulk-scale intergranular stresses caused by gravity. Here I express this ratio as:

$$S = \frac{\rho_s \dot{\gamma}^2 \delta^2}{\sigma_e} \quad , \tag{2}$$

where $\dot{\gamma}$ is the characteristic shear rate and σ_e is a characteristic value of the intergranular effective normal stress. The effective stress is defined as $\sigma_e = \sigma - p$, where σ is the total normal stress in the mixture and p is the pore-fluid pressure. Most definitions of S use σ in place of σ_e because they apply to dry grain flows, but here I use σ_e because pore fluid can significantly affect grain-contact stresses in debris flows. Considerable recent work relating the small-scale and large-scale mechanics of dense granular flows emphasizes the importance of S, but does so by employing its square root, called the "inertia number" [15,18].

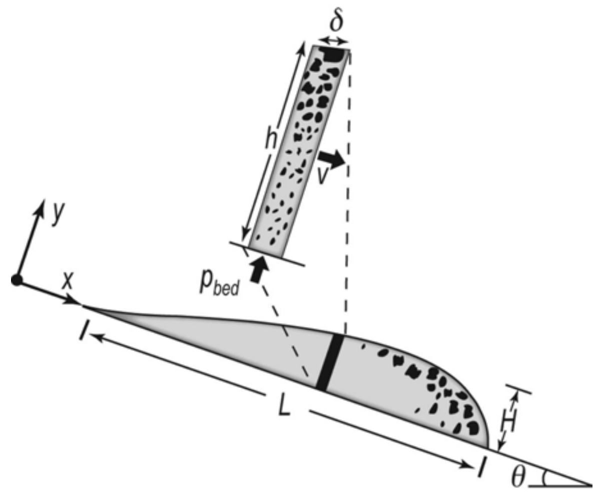

FIGURE 1. Schematic illustrating coordinate system and basic attributes of a debris-flow surge descending a uniform slope.

MATHEMATICAL MODEL

The model emphasizes motion of the granular solid phase and treats fluid flow in a frame of reference that moves with the solids. Formally, this approach is predicated on the assumption that fluid flow relative to the solids is very slow relative to $\vec{v}_s$, the velocity of the solids themselves [6]. Thus, if the fluid's velocity in a fixed reference frame is $\vec{v}_f$, then its apparent velocity (*i.e.*, specific discharge or volume flux per unit area) relative to the solids is $\vec{q} = (\vec{v}_f - \vec{v}_s)(1-m)$, and the model assumes that $\|\vec{q}\| << \|\vec{v}_s\|(1-m)$. On the other hand, conservation of mass requires that:

$$\nabla \bullet \vec{v}_s = -\nabla \bullet \vec{q} \quad . \tag{3}$$

The left-hand side of (3) plays a key role in evaluating depth-integrated bulk mass conservation, and the right-hand side plays a crucial role in evaluating inter-phase drag and attendant pore-pressure evolution.

Depth-Averaged Conservation Laws

The depth-integrated mass-conservation equation for a grain-fluid mixture with variable m may be expressed as [19]:

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} &= \int_0^h (\nabla \bullet \vec{v}_s) dy \\ &= -\int_0^h \frac{1}{m}\frac{dm}{dt} dy = D \quad , \end{aligned} \tag{4}$$

where $d/dt = \partial/\partial t + \vec{v}_s \bullet \nabla$ is a material time derivative in a frame of reference that moves with the granular phase, and D summarizes the depth-integrated granular dilation rate. The net specific discharge of fluid through the free surface at y=h equals −D, because (3) and (4) together imply that $D = -\int_0^h (\nabla \cdot \vec{q})dy$. At the surface of debris flows such discharge is seldom detectable, but pore-pressure gradients associated with $\vec{q}$ can be large even when the discharge is negligible. (See the section below on *Basal Pore-fluid Pressure*.) The model's mass and momentum conservation equations assume that any discharged fluid leaves the flow but remains available for reincorporation.

The depth-integrated x-direction momentum-conservation equation for the grain-fluid mixture may be expressed as [19]:

$$\begin{aligned} &\frac{\partial(hv)}{\partial t} + \frac{\partial(hv^2)}{\partial x} = h\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right] + vD \\ &= gh\sin\theta - \kappa gh\cos\theta\frac{\partial h}{\partial x} - (1-\kappa)\frac{h}{\rho}\frac{\partial p_{bed}}{\partial x} - \frac{\tau}{\rho}, \end{aligned} \tag{5}$$

where p_{bed} is the basal pore-fluid pressure, κ is a solid-phase longitudinal normal-stress coefficient (of order 1) that depends on granular friction [6,7,8,9,11], and τ is the basal shear traction resisting downslope motion. On the first line of (5), the term vD accompanies $h[\partial v/\partial t + v(\partial v/\partial x)]$ because it accounts for the effects of mass flux through the free surface that occurs when D is nonzero.

The derivation of (5) involves some assumptions closely analogous to those of standard shallow-water theory [9]. Specifically, it assumes that there is negligible differential advection of x-direction momentum (caused by variation of velocity as a function of y), and it assumes that p varies linearly as a function of y (with $p = p_{bed}$ at $y = 0$ and $p = 0$ at $y = h$). Errors resulting from these assumptions can be eliminated by inserting appropriate numerical correction factors (of order 1) in the terms containing $\partial v/\partial x$ and $\partial p_{bed}/\partial x$. Such correction factors play no fundamental role in development of the model, however, and I omit them here for the sake of clarity.

Equation (5) reduces to the standard momentum equation used in shallow-water theory if ρ is constant and the granular mass is completely liquefied by high pore-fluid pressure, such that $p_{bed} = \rho gh\cos\theta$ applies. On the other hand, it reduces to the Savage-Hutter granular avalanche model [7] if $p_{bed} = 0$. If $\kappa = 1$ applies, the granular-avalanche and shallow-water models collapse to the same form, but in general κ may vary from about 1/4 to 4 in deforming granular materials. (See the section below titled *Longitudinal Normal Stress Coefficient*.) As a result, the effects of variations in longitudinal normal stress described by (5) can be more complex than those described by standard shallow-water theory.

Constitutive Relations

Although (4) and (5) govern evolution of v and h, and (1) provides a formula for ρ, these equations also contain m, p_{bed}, τ, and κ, which are as yet undetermined. Evaluation of these quantities (as well as σ_e and the dilatancy angle ψ that relates volume change to shearing) involves a combination of physical reasoning and constitutive postulates.

Volume Fraction and Dilatancy

The solid volume fraction m and dilatancy angle ψ evolve in an interdependent way, because volume-fraction change depends, in part, on dilation caused by shearing, while the dilatancy angle itself depends on the ambient volume fraction relative to a volume fraction that is equilibrated to the current shear rate and state of stress. The depth-integrated dilation rate D (defined kinematically in (4)) results from mechanical processes summarized by:

$$D = v\tan\psi - \int_0^h \alpha\frac{d\sigma_e}{dt}dy \quad . \tag{6}$$

Here α is the debris' poroelastic compressibility, and the term containing α describes the depth-integrated rate of compression (negative dilation) caused by changes in effective stress, σ_e. This term can be nonzero even if $v = 0$. On the other hand, the term $v\tan\psi$ describes the rate of dilation caused by shearing at the depth-integrated rate $\int_0^h \dot{\gamma}dy = v$. The dilatancy angle ψ can be positive, negative, or zero, depending on the current value of the volume fraction m.

An additional constitutive equation is required to link the current value of ψ to the current value of m. To establish this linkage I employ a rationale similar to that in [14], and postulate that ψ satisfies:

$$\begin{aligned} \tan\psi &= \frac{1}{h}\int_0^h C_1[m - m_{eq}]dy \\ &= \frac{1}{h}\int_0^h C_1[m - (m_{crit} - C_2 S)]dy, \end{aligned} \tag{7}$$

where C_1 and C_2 are positive constants to be determined by experimentation or calibration, m_{eq} is the solid volume fraction in equilibrium with the state represented by S, and m_{crit} is the value of m_{eq} that applies in a static critical state with $S = 0$ and ambient effective stress σ_e. To define S in (7), I use the depth-averaged value $\dot{\gamma} = v/h$ in (2), yielding:

$$S = \frac{\rho_s v^2}{\sigma_e}\left(\frac{\delta}{h}\right)^2 \quad . \tag{8}$$

Taken together, (6), (7) and (8) describe simultaneous evolution of ψ, m, and S in response to evolution of v, h, and σ_e, but they also can be combined with the definition of D in (4) to form a single differential equation describing evolution of m:

$$\frac{dm}{dt} = -C_1 \frac{v}{h} m[m - (m_{crit} - C_2 S)] + m\alpha \frac{d\sigma_e}{dt} . \tag{9}$$

Although the model ultimately treats m and σ_e as depth-averaged quantities, (9) applies even if they are not depth-averaged. The difference between the depth-averaged and depth-dependent interpretations of (9) hinges on the interpretation of the material time derivatives dm/dt and $d\sigma_e/dt$. The depth-averaged interpretation assumes that $\partial m/\partial y = \partial \sigma_e/\partial y = 0$, such that dm/dt reduces from $\partial m/\partial t + \vec{v}_s \bullet \nabla m$ to $\partial m/\partial t + v\,\partial m/\partial x$, and likewise for $d\sigma_e/dt$.

Basal Pore-fluid Pressure

To evaluate p_{bed} in (5), I first consider $p(y)$ and decompose it into a hydrostatic component that balances the pore-fluid weight and a nonhydrostatic or "excess" component, denoted by p':

$$p(y) = \rho_f g(h - y)\cos\theta + p'(y) \quad . \tag{10}$$

In an ideal steady flow, $p' = 0$; otherwise dilation or contraction produces nonzero p', and evolution of p' is coupled to evolution of m.

Following [6], I postulate that the gradient of p' depends linearly on the specific discharge of fluid relative to the granular solids. That is, I use a Darcian drag rule:

$$\vec{q} = -\frac{k(m)}{\mu}\nabla p' \quad , \tag{11}$$

where $k(m)$ is the hydraulic permeability of the granular aggregate, which decreases as m increases, and μ is the pore-fluid viscosity, assumed to be constant. For debris-flow materials, values k~10^{-11} m^2 and μ ~0.1 Pa-s are typical [6], implying that values of $\vec{q}$ associated with reasonable values of $\nabla p'$ (which are no larger than $\rho\vec{g}$) are smaller than 10^{-5} m/s. The great dissimilarity of this velocity and typical debris-flow velocities (~10 m/s) provides justification for analyzing fluid flow in a frame of reference that moves with the granular solids and for neglecting fluid flux through the free surface at $y = h$.

A general equation describing evolution of p' results from substitution of (11) into (3) and then (3) into (4):

$$D = -\int_0^h \frac{1}{m}\frac{dm}{dt}dy = \int_0^h \left(\nabla \bullet \frac{k}{\mu}\nabla p'\right)dy . \tag{12}$$

According to (12), p' evolves diffusively in response to changes in m, but only the depth integral of the diffusive flux is relevant with respect to the depth-integrated dilation rate, D.

To simplify evaluation the depth integral on the right-hand side of (12), I assume that k itself is a depth-integrated parameter and then invoke a scaling argument like that used in [11]. Because the length scale in the x direction is L and the length scale in the y direction is H (Figure 1), it follows that $\partial/\partial x^2 \sim 1/L^2$, whereas $\partial/\partial y^2 \sim 1/H^2$. Thus, because $H<<L$ is typical, I infer that $\partial^2 p'/\partial x^2 << \partial^2 p'/\partial y^2$, leading to the approximation $(k/\mu)(\partial^2 p'/\partial y^2)$ for the integrand on the right-hand side of (12). Evaluation of the integral then reduces (12) to

$$D = \frac{k}{\mu}\frac{\partial p'}{\partial y}\bigg|_{y=h} \quad . \tag{13}$$

This equation does not include $(k/\mu)[\partial p'/\partial y]_{y=0}$ because the no-flux basal boundary condition stipulates that $[\partial p'/\partial y]_{y=0} = 0$. Thus, (13) uniquely relates the gradient of p' at the free surface to D, but it provides no explicit information on the desired quantity, p'_{bed}.

To obtain an equation for p'_{bed}, I invoke an additional boundary condition and use a polynomial approximation of a pore-pressure diffusion solution. The boundary condition $p'(h) = 0$ specifies that no excess pore pressure exists at the free surface, because the pressure there remains atmospheric. Consequently, $p'(y)$ must satisfy this condition along with (13) and $[\partial p'/\partial y]_{y=0} = 0$. Moreover, $p'(y)$ should mimic pore-pressure distributions associated with diffusive fluxes. A distribution that satisfies all these criteria is given by [11]:

$$p'(y) = -\frac{\mu}{k}Dh\left[\left(1 - \frac{y}{h}\right) - \frac{1}{6}\left(1 - \frac{y}{h}\right)^6\right] . \tag{14}$$

Substitution of (14) in (10) produces a profile of $p(y)$ that differs only slightly from the linear profile assumed in the derivation of (5).

Evaluation of (10) and (14) at y=0 provides an equation for p_{bed}:

$$p_{bed} = \rho_f gh\cos\theta - \frac{5}{6}\frac{Dh\mu}{k}. \tag{15}$$

Although (15) does not indicate any explicit time-dependence, time-dependence of p_{bed} is implicit in its dependence on the evolving values of *h* and *D*.

Alternative expressions for p_{bed} result if a distribution $p'(y)$ different from that given by (14) is present. Physically reasonable alternatives must satisfy (13) and the relevant boundary conditions, however, and such alternatives involve no change in the general form of (15) and only a modest change in the numerical coefficient 5/6. Therefore, I infer that (15) is a suitable approximation.

Lower and upper bounds exist for viable values of p_{bed} predicted by (15). For example, if dilation occurs rapidly enough that $5D\mu/[6k\rho_f g\cos\theta] > 1$, (15) predicts that $p_{bed} < 0$. This result implies that cavitation of pore fluid occurs, violating the assumption that the debris is fully saturated with liquid. On the other hand, if negative dilation (i.e., contraction) occurs rapidly enough that $5D\mu/[6k(\rho-\rho_f)g\cos\theta] < -1$, (15) predicts that p_{bed} more than suffices to counteract the weight of the overlying debris. This result implies that unbalanced forces exist in the *y* direction, violating another model assumption. Although these violations are not prohibited by physical phenomena, computationally it may be advisable to limit p_{bed} to the range $0 \le p_{bed} \le \rho gh\cos\theta$ to avert inconsistencies in solutions. Investigation of the need for such a stop-gap approach awaits numerical implementation of the model.

Effective Normal Stress

Following the usual conventions of shallow-flow theory, I assume that the total normal stress in the *y* direction, σ, is due simply to the bed-normal component of the weight of the overlying debris:

$$\begin{aligned}\sigma(y) &= \rho g(h-y)\cos\theta \\ &= [\rho_s m + \rho_f(1-m)]g(h-y)\cos\theta.\end{aligned} \tag{16}$$

A more complete formulation would include modification of the weight as a consequence of debris acceleration normal to the bed [20], but as is customary in most shallow-flow theories, I assume that such acceleration is negligible.

An equation describing the effective normal stress $\sigma_e = \sigma - p$ is obtained by combining (16) with (10), yielding:

$$\begin{aligned}\sigma_e(y) &= \rho g(h-y)\cos\theta - p(y) \\ &= [\rho_s - \rho_f]mg(h-y)\cos\theta - p'(y).\end{aligned} \tag{17}$$

The value of σ_e at $y = 0$ has particular importance in the model because it affects the basal shear traction, τ. By combining (15) and (17), this basal value of σ_e may be expressed as:

$$\begin{aligned}\sigma_{e\,bed} &= \rho gh\cos\theta - p_{bed} \\ &= [\rho_s - \rho_f]mgh\cos\theta + \frac{5}{6}\frac{Dh\mu}{k}.\end{aligned} \tag{18}$$

Although (18) does not demonstrate any explicit time-dependence, $\sigma_{e\,bed}$ has implicit time-dependence through its dependence on *m*, *h* and *D*.

Basal Shear Traction

Following the rationale in [6], I assume that the basal shear traction (sometimes called basal shear stress) τ obeys the Coulomb friction rule, modified to account for the effect of basal pore-fluid pressure:

$$\tau = \sigma_{e\,bed}\tan\phi_{bed}\quad, \tag{19}$$

where ϕ_{bed} is the Coulomb friction angle of the granular phase in contact with the bed. Equation (19) could be generalized to include dependence of ϕ_{bed} on evolving values of *S*, ψ, or perhaps other quantities [14,15,18], but here, for the sake of parsimony, I treat ϕ_{bed} as a constant. Even with a constant value of ϕ_{bed}, however, τ can express rate-dependent flow resistance owing to its dependence on $\sigma_{e\,bed}$. This rate-dependent effect arises from the rate-dependent component of p_{bed}, which is $\sim Dh\mu/k$ as shown in (18). The presence of even a modest dilation rate *D* can result in considerable rate-dependence owing to the typical values $k \sim 10^{-11}$ m^2 and $\mu \sim 0.1$ Pa-s.

Deformation of the pore fluid associated with the bulk shear rate $\dot\gamma$ could also produce a rate-dependent contribution to τ (i.e., $\sim \mu\dot\gamma$). Generally, however, such shear resistance is minimal in comparison to granular shear resistance in debris flows [21], and here, in keeping with the goal of parsimony, I omit it.

A final important point regarding (19) is that, although the Coulomb friction rule is classically a quasi-static relationship, a Coulomb-like proportionality between normal and shear stress also applies in rapidly shearing mixtures of grains and fluid. Bagnold [22] identified this proportionality, and his central findings were later duplicated and articulated more thoroughly by others [23]. A current consensus is that (19) applies in most dense, granular shear flows, although ϕ_{bed} may vary as a function of S or some other measure of shear intensity [15,18].

Longitudinal Normal Stress Coefficient

The final quantity requiring evaluation is κ, a proportionality coefficient that relates the longitudinal normal stresses communicated by the solid grains, σ_x to the effective intergranular normal stress σ_e:

$$\kappa = \frac{1}{h}\int_0^h \frac{\sigma_x}{\sigma_e} dy \quad . \tag{20}$$

Here the integral demonstrates that κ is inherently a depth-averaged quantity. If $\kappa = 1$, then the depth integral of the longitudinal normal stress gradient is $\rho g h(\partial h / \partial x)$ and the implied stress state is hydrostatic, as assumed in conventional shallow-water theory. On the other hand, if $\kappa \neq 1$ the longitudinal stress gradient differs.

I obtain the value of κ by following the reasoning originally employed in the Savage-Hutter granular avalanche theory [7,8], in which a Mohr's stress diagram is used to derive κ as a function of ϕ_{bed} and ϕ_{int}, the Coulomb friction angle for internal shearing. For shearing that occurs both internally and along the bed, the Mohr's construction implies [6]:

$$\kappa = 2\frac{1 \mp [1 - \cos^2\phi_{\text{int}}(1 + \tan^2\phi_{bed})]^{1/2}}{\cos^2\phi_{\text{int}}} - 1, \tag{21}$$

where the "−" in $\mp$ applies to "active" or extensional longitudinal deformation in which $\partial v / \partial x > 0$, whereas the "+" in $\mp$ applies to "passive" or compressional longitudinal deformation in which $\partial v / \partial x < 0$. If $\partial v / \partial x = 0$, then the value of κ is indeterminate, although $\kappa = 1$ is perhaps a reasonable assumption.

Once an appropriate value of κ is obtained, the depth average of σ_x is expressed simply as the depth average of $\kappa\sigma_e$, which is approximated very closely by $(1/2)\kappa\sigma_{e\,bed}$. A more exact value can be obtained by integrating (17) to find its depth average, but this step is probably unwarranted because (17) is itself an approximation owing to its dependence on (14).

EXACT SOLUTIONS OF SIMPLIFIED EQUATIONS

Some important physical implications of the model are revealed by considering hypothetical special cases. One significant special case assumes that $\psi = 0$. Then the model couples evolution of h and v to evolution of p_{bed} through use of a depth-averaged diffusion equation [11]. Here I describe two additional special cases, for which analytical solutions demonstrate implications of $\psi \neq 0$.

Accelerating Slab with Constant Dilatancy

If all variations with respect to x are neglected, the model describes downslope motion of a uniform slab of debris that is free to accelerate. Furthermore, if dilation is due to a constant ψ, such that dependence of ψ on σ_e and m is neglected, then the governing equations (4), (5), and (9) reduce to:

$$\frac{dh}{dt} = v \tan\psi \ , \tag{22}$$

$$\frac{dm}{dt} = -\frac{m}{h} v \tan\psi \ , \tag{23}$$

$$\begin{aligned} \frac{dv}{dt} = {} & g\sin\theta - \frac{(\rho_s - \rho_f)m}{\rho} g\cos\theta\tan\phi_{bed} \\ & - v\frac{5}{6}\frac{\mu\tan\psi}{k\rho}\tan\phi_{bed} - v^2\frac{\tan\psi}{h} \ . \end{aligned} \tag{24}$$

Equations (22) and (23) show that the thickness h and volume fraction m of the slab evolve in a simple manner that depends on the dilation rate $v\tan\psi$. Equation (24), on the other hand, shows that evolution of v depends on a balance between the gravitational driving term $g\sin\theta$ and three resisting terms with distinct physical origins. The first resisting term arises from the product of the y component of the buoyant weight of the debris and basal friction coefficient, $\tan\phi_{bed}$. The second resisting term contains v and arises from modification of basal friction by excess pore pressure generated by dilation. The third resisting term contains v^2 and arises from the mass and momentum change associated with dilation (i.e., the term vD in (5)).

As a consequence of its three resisting terms, (24) superficially resembles debris-flow momentum equations that use shear traction formulas of the form $\tau = a_0 + a_1 v + a_2 v^2$, where the coefficients a_0, a_1, and a_2 are treated as calibration parameters [24,25]. In contrast, values of the analogous coefficients in (24) are derived using physical arguments. The coefficient values can vary moderately in (24) but can vary greatly if ψ is allowed to vary as in (7).

A key insight to model behavior is gained by considering a simpler form of (24) that results from assuming that ψ is small (<0.01) and that m and h are constant. Then, normalizing (24) using $v^* = v/\sqrt{gL}$ and $t^* = t/\sqrt{L/g}$ and retaining only those terms likely to be of order 1 or larger reduces the equation to

$$\frac{dv^*}{dt^*} = A - Bv^* , \qquad (25)$$

where A and B are constants defined by

$$A = \sin\theta - \frac{(\rho_s - \rho_f)m}{\rho}\cos\theta \tan\phi_{bed} , \qquad (26)$$

$$B = \frac{5}{6}\left[\frac{\sqrt{L/g}}{(k/\mu)\rho}\right]\tan\psi \tan\phi_{bed} . \qquad (27)$$

A solution of (25) satisfying the initial condition $v^*(0) = 0$ is $v^* = (A/B)[1-\exp(-Bt^*)]$. This result implies that if $B>0$, v^* approaches the stable, steady value $v^* = A/B$ as time proceeds, but if $B<0$, v^* grows unstably.

The divergent behaviors of solutions of (25) indicate that the phenomena determining the sign and magnitude of B have great physical importance. The sign of B is determined by the sign of $\tan\psi$, such that positive dilation leads to stable, steady motion regulated by negative feedback associated with viscous pore-fluid flow. On the other hand, negative dilation (contraction) produces unstable, positive feedback comparable to that of a negative viscosity coefficient.

The magnitude of feedback effects in (25) is determined mostly by the term in brackets in (27), which may be viewed as a timescale ratio. The numerator of this ratio is the timescale for debris dilation, and the denominator is the timescale for modification of effective basal normal stress by excess pore-fluid pressure caused by dilation. Typical values of this timescale ratio are >>1 as a consequence of very small values of k (typically < 10^{-8}m^2), implying that feedback caused by pore-pressure change can be very strong. Indeed, such feedback might overwhelm all other aspects of debris-flow dynamics were it not for the fact that ψ itself evolves.

Steady, Uniform Flow with Evolving Dilatancy

The physical implications of an equation similar to (9), which governs evolution of m in the presence of evolving ψ , have been discussed previously [14]. These implications can be illustrated by considering an imaginary case of steady, uniform flow in which dilation occurs but v, h, and σ_e are constant. Then $m_{eq} = m_{crit} - C_2 S$ is also constant, and use of this expression in (9) reduces the equation to

$$\frac{dm}{dt} = -Cm(m - m_{eq}) \qquad (28)$$

where C is a constant defined as

$$C = C_1 \frac{v}{h} \qquad (29)$$

A solution of (28) satisfying the initial condition $m(0) = m_0$ is

$$\frac{m}{m_{eq}} = \left[1 - \left(1 - \frac{m_{eq}}{m_0}\right)\exp(-m_{eq}Ct)\right]^{-1} \qquad (30)$$

which implies that m relaxes toward m_{eq} with a characteristic time $1/m_{eq}C$. Moreover, as $m \to m_{eq}$, it follows that $\psi \to 0$, leading ultimately to a perfect steady state in which m is in equilibrium with the ambient value of S. This result shows that the unconditional stability or instability of solutions of (25) tell only a fraction of the story implied by the model equations. Better understanding of the full implications awaits numerical computations.

CONCLUDING DISCUSSION

The new model I propose attempts to remedy a fundamental deficiency of previous depth-averaged debris-flow models. It does so by including the effects of evolving volume fractions and dilatancy, along with consequent pore-pressure feedbacks. Inclusion of these effects requires use of three constitutive parameters not used in previous debris-flow models, m_{crit} , C_1, and C_2. These parameters appear in equation (7), which relates the volume fraction, m, dilatancy, ψ , and Savage number, S. Thus, (7) may be regarded as the principal new postulate in the model. It is consistent with concepts of critical-state soil mechanics as previously employed elsewhere [14].

A shortcoming of the new model is its lack of explicit accounting for grain-size segregation. Such segregation is pervasive in debris flows, and it leads to development of coarse-grained snouts and lateral margins with δ and k values much larger than those of adjacent, finer-grained debris [6,21]. Representation of segregation effects therefore requires manipulation of δ and k values, whereas a more satisfactory model would describe the process of segregation.

As an aid to numerical implementation, Tables 1-3 summarize the model's symbols and their meanings. The model's governing differential equations are (4), (5), and (9), which describe simultaneous evolution of h, v, and m (Table 1). These equations may all be expressed in a Lagrangian form that employs the material time derivative $d/dt = \partial/\partial t + v(\partial/\partial x)$, and as a result, they can be solved using either an Eulerian or Lagrangian numerical method. A Lagrangian formulation offers advantages because embedded within (4), (5), and (9) are eight functions describing

TABLE 1. Dependent variables that evolve as functions of *x* and *t*.

Variable [Dimensions]	Symbol	Equation
Flow thickness [L]	h	4
Depth-averaged flow velocity [L/T]	v	5
Depth-averaged solid volume fraction [0]	m	9

TABLE 2. Quantities that evolve as specified functions of *h*, *v*, and *m*. All quantities are depth-averaged unless otherwise indicated.

Quantity [Dimensions]	Symbol	Equation
Debris bulk density [M/L^3]	ρ	1
Dilation rate [L/T]	D	6
Dilatancy angle [0]	ψ	7
Savage number [0]	S	8
Basal pore pressure [M/LT^2]	p_{bed}	15
Basal effective normal stress [M/LT^2]	$\sigma_{e\ bed}$	17
Basal shear traction [M/LT^2]	τ	19
Longitudinal normal stress coefficient [0]	κ	21

TABLE 3. Model parameters with specified values. Values are depth-averaged unless otherwise indicated.

Quantity	Symbol	Typical values
Solid grain density	ρ_s	2500-2800 kg/m^3
Pore-fluid density	ρ_f	1000-1200 kg/m^3
Gravitational acceleration magnitude	g	9.8 m/s^2
Slope angle	θ	5-45 degrees
Basal friction angle	ϕ_{bed}	20-45 degrees
Internal friction angle	ϕ_{int}	20-45 degrees
Characteristic grain diameter	δ	0.01-1 m
Hydraulic permeability	k	10^{-15}-10^{-8} m^2
Pore-fluid viscosity	μ	0.001-1 Pa-s
Debris compressibility	α	10^{-5}-10^{-4} Pa^{-1}
Critical-state value of *m*	m_{crit}	0.5-0.7
Dilation coefficient 1	C_1	~1
Dilation coefficient 2	C_2	~0.1

evolution of material or kinematic properties that advect with the moving debris (Table 2). Taken together, the relationships summarized in Tables 1 and 2 describe a system of 11 simultaneous equations containing 11 unknowns, all of which involve a material frame of reference.

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